# Lie algebra classification, conservation laws and invariant solutions for the a particular case of the generalized Levinson-Smith equation

## Clasificación del álgebra de Lie, leyes de conservación y soluciones invariantes para un caso particular de la ecuación generalizada e Levinson-Smith

G. I. Loaiza-Ossa ; Y. A. Acevedo-Agudelo ; O. M. Londoño-Duque ; D. A. García-Hernández

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***Abstract*—In this study, we examine a specific instance of the generalized Levinson-Smith equation, which is linked to the Liènard equation and holds significant importance from the perspectives of physics, mathematics, and engineering. This underlying equation has practical applications in mechanics and nonlinear dynamics and has been extensively explored in the qualitative scheme. Our approach involves applying the Lie group method to this equation. By doing so, we derive the optimal generating operators for the system that pertain to the specific instance of the generalized Levinson-Smith equation. These operators are then used to define all invariant solutions associated with the equation. In addition, we demonstrate the variational symmetries and corresponding conservation laws using Noether's theorem. Finally, we categorize the Lie algebra related to the given equation.**

***Index Terms*— Conservation laws; Invariant solutions; Lie algebra classification; Lie symmetry group; Noether's theorem; Optimal system; Variational symmetries.**

***Resumen*—En este estudio, examinamos una instancia específica de la ecuación generalizada de Levinson-Smith, que está vinculada con la ecuación de Liènard y tiene una gran importancia desde las perspectivas de la física, las matemáticas y la ingeniería. Esta ecuación subyacente tiene aplicaciones prácticas en mecánica y dinámica no lineal, y ha sido ampliamente explorada en el esquema cualitativo. Nuestro enfoque implica aplicar el método de grupos de Lie a esta ecuación. Al hacerlo, obtenemos los operadores generadores óptimos del sistema que se refieren a la instancia específica de la ecuación generalizada de Levinson-Smith. Luego, se utilizan estos operadores para definir todas las soluciones invariantes asociadas con la ecuación. Además, demostramos las simetrías variacionales y las leyes de conservación correspondientes utilizando el teorema de Noether. Finalmente, categorizamos el álgebra de Lie relacionada con la ecuación dada.**

***Palabras claves*— Clasificación del Álgebra de Lie; Grupo de simetrías de Lie; Leyes de Conservación; Simetrías Variacionales; Sistema óptimo; Soluciones invariantes; Teorema de Noether.**

### I. Introduction

THE Lie group symmetry method is a powerful mathematical tool employed to investigate a wide range of differential equations, including ODEs, PDEs, FPDEs, and FODEs. This mathematical theory was first introduced in the 19th century by Sophus Lie, [1], following the principles of Galois Theory in algebra. The Lie group method has attracted considerable interest among researchers in various branches of science, such as mathematics, theoretical and applied physics, due to the physical interpretations that can be derived from the underlying equations being studied. As a result, this method enables the construction of conservation laws, utilizing the celebrated Noether's theorem [2], and similarity solutions, which are not achievable using traditional methods, particularly when utilizing Ibragimov's approach [3].

Furthermore, this method has contributed to establishing frameworks and the efficacy of certain numerical methods, leading to the development of numerous software packages across various computational environments, as exemplified by [4, 5]. In general, due to the importance of studying equations such as ODEs, PDEs, and others, the Lie group method is of interest to a diverse range of scientists. The literature contains a vast array of references on the Lie group method, including [6, 7, 8, 9]. Recently, the Lie group method has been employed to solve and analyze various problems in numerous scientific fields. For example, in [10], a model with applications in quantum field and differential geometry theory was studied using this method. Moreover, [11, 12, 13, 14, 15, 16, 17, 18]

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G. I. Loaiza-Ossa. Full-time professor at Universidad EAFIT, Colombia, e-mail: gloaiza@eafit.edu.co.

Y. A. Acevedo-Agudelo. Master's Degree in Applied Mathematics, Universidad EAFIT, Colombia. e-mail: yaceved2@eafit.edu.co.

O. M. Londoño-Duque. PhD of applied mathematics, University of Campinas, Brazil. e-mail: olondon2@eafit.edu.co.

D. A. García-Hernández. PhD in Mathematics, University of Campinas, Brazil. e-mail: d190684@dac.unicamp.br

provide references on the latest advancements in symmetry analysis.

The equation $y_{xx}+\phi(x,y,y_x)y_x=\gamma(x,y)$, which belongs to the class of generalized Levinson-Smith equations, is closely connected to the second-order nonlinear differential equation of the Liènard type [19]. These equations form the foundation for a wide range of phenomena across various disciplines, including mechanics, electronics, biology, seismology, chemistry, and physics. For example, an important model used in physical is the Van der Pol equation, which describes a non-conservative oscillator with non-linear damping.

In their work [20], Levinson and Smith investigated a generalized equation that describes relaxation oscillations. The study of such oscillations is fundamental to understanding a variety of phenomena in fields ranging from electronics and physics to biology and chemistry. By examining this equation in depth, Levinson and Smith provided insights into the dynamics of relaxation oscillations and their role in complex systems.

Within this paper, we examine the subsequent differential equation (1):

$$y_{xx}+\phi(x,y,y_x)y_x=\gamma(x,y). \quad (1)$$

The differential equation presented in (1), indicates that the friction coefficient, represented by the function $\phi$, is dependent upon the variables $x,y$, and $y_x$, and is typically a non-linear function. Additionally, the function $\gamma$, which is referred to as the disturbance function, is also non-linear. It's important to note that (1) is classified as a generalization of the Levinson-Smith equation. In, Duarte et al, introduce a particular case of (1)

$$y_{xx}=-2y_x^2(x+y)^{-1}-y_x(x+y)^{-1}, \quad (2)$$

and its solution (3):

$$y(x)=-\frac{x^2+2C_1}{2x+2C_2}, \text{ where } C_2, C_1 \text{ are constants.} \quad (3)$$

In this work, they present the Lie group of symmetries of (2), using a ODEtools Maple package. thus, the proposal of our work is: $i$) to calculate the $5-$dimensional Lie symmetry group in all detail, $ii$) to present the optimal system (optimal algebra) for (2), $iii$) making use of all elements of the optimal algebra, to propose invariant solutions for (2), then $iv$) to construct the Lagrangian with which we could determine the variational symmetries using Noether′s theorem, and thus to present conservation laws associated, and finally $v$) to classify the Lie algebra that is associated with (2), and corresponds to the Lie symmetry group.

To conclude this initial section, it is crucial to highlight that the generalized Levinson-Smith equation is a non-linear partial differential equation utilized in the elasticity theory to depict the seismic wave propagation within porous media. This equation serves as a model for the propagation of seismic waves through porous media such as soils, fractured rocks, and petroleum reservoirs. The equation's solution offers scientists and engineers a better comprehension of the seismic wave diffusion across these porous media, which is critical for natural resource exploration and production, geotechnical engineering, and seismic risk assessment.

## II. Continuous Lie Symmetry Groups

In this section, we analyze the Lie symmetry group of (2). The principal outcome of this section may be formulated as follows:

Proposition 1. The set of vector fields that generate the Lie group of symmetries for (2) is given by:

$$\Pi_1=-(x+y)^{-1}\frac{\partial}{\partial x},\quad \Pi_2=-y(x+y)^{-1}\frac{\partial}{\partial x},\quad \Pi_3=x(x+2y)(x+y)^{-1}\frac{\partial}{\partial x},$$
$$\Pi_4=-\frac{\partial}{\partial x}+\frac{\partial}{\partial y},\quad \text{and} \quad \Pi_5=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}. \quad (4)$$

Proof. The general form for the generator operators of a Lie group of a parameter admitted by (2) is:

$$x\to x+\epsilon\xi(x,y)+O(\epsilon^2), \quad \text{and} \quad y\to y+\epsilon\eta(x,y)+O(\epsilon^2),$$

where $\epsilon$ is the group parameter. The vector field associated with this group of transformations is $\Gamma=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y}$, with $\xi,\eta$ differentiable functions in $\mathbb{R}^2$. To find the infinitesimals $\xi(x,y)$ and $\eta(x,y)$, we applied the second extension operator:

$$\Gamma^{(2)} = \Gamma+\eta_{[x]}\frac{\partial}{\partial y_x}+\eta_{[xx]}\frac{\partial}{\partial y_{xx}}, \quad (5)$$

After applying (5) to (2), we obtain the following symmetry condition:

$$\xi(-2y_x^2(x+y)^{-2}-y_x(x+y)^{-2})+\eta(-2y_x^2(x+y)^{-2}-y_x(x+y)^{-2}) + \eta_{[x]}(4y_x(x+y)^{-1}+(x+y)^{-1})+\eta_{[xx]}=0, \quad (6)$$

where $\eta_{[x]}$ and $\eta_{[xx]}$ are the coefficients in $\Gamma^{(2)}$ given by (see [21, 22]):

$$\begin{aligned}\eta_{[x]} &= D_x[\eta]-(D_x[\xi])y_x=\eta_x+(\eta_y-\xi_x)y_x-\xi_y y_x^2.\\ \eta_{[xx]} &= D_x[\eta_{[x]}]-(D_x[\xi])y_{xx},\\ &= \eta_{xx}+(2\eta_{xy}-\xi_{xx})y_x+(\eta_{yy}-2\xi_{xy})y_x^2-\xi_{yy}y_x^3\\ &\quad +(\eta_y-2\xi_x)y_{xx}-3\xi_y y_x y_{xx}. \quad (7a)\end{aligned}$$

where $D_x$ is the total derivative operator: $D_x=\partial_x+y_x\,\partial_y+y_{xx}\,\partial_{y_x}+\cdots$.

After applying (7a) in (6) and substitute in the resulting expression $y_{xx}$ by (2), is obtained:

$$\begin{aligned}&(2\xi_y(x+y)^{-1}-\xi_{yy})y_x^3+(\eta_{yy}-2\xi_{xy}-2\xi(x+y)^{-2}-2\eta(x+y)^{-2}\\ &+2\eta_y(x+y)^{-1}+3\xi_x(x+y)^{-1}+2\xi_y(x+y)^{-1})y_x^2\\ &+(-\xi(x+y)^{-2}-\eta(x+y)^{-2}+4\eta_x(x+y)^{-1}+\xi_x(x+y)^{-1}+2\eta_{xy}-\xi_{xx})y_x\\ &+\eta_x(x+y)^{-1}+\eta_{xx}=0. \quad (7b)\end{aligned}$$

Analyzing the coefficients in (7b) with respect to the independent variables $y_x^3, y_x^2, y_x, 1$ we get the following system of determining equations, with $(x+y)\neq 0$:

$$2\xi_y - \xi_{yy}(x+y) = 0, \quad (8a)$$
$$(\eta_{yy} - 2\xi_{xy})(x+y)^2 - 2\xi - 2\eta + (2\eta_y + 3\xi_x + 2\xi_y)(x+y) = 0, \quad (8b)$$
$$-\xi - \eta + (4\eta_x + \xi_x)(x+y) + (2\eta_{xy} - \xi_{xx})(x+y)^2 = 0, \quad (8c)$$
$$\eta_x + \eta_{xx}(x+y) = 0. \quad (8d)$$

Solving the system in (8a)-(8d), for $\xi$ and $\eta$ we get

$$\xi = -k_1(x+y)^{-1} - k_2 y(x+y)^{-1} + k_3 x(x+2y)(x+y)^{-1} - k_4 + k_5 x,$$
$$\eta = k_4 + k_5 y. \quad (8e)$$

Note that in (8e), $k_1, k_2, k_3, k_4$ and $k_5$ are arbitrary constants. Thus, the generators of the group of symmetries of (2) are the operators $\Pi_1 - \Pi_5$ described in the statement of the Proposition 1; thus achieving the proposed result. □

## III. Optimal Algebra

Taking into account [23, 24, 25, 26], we present in this section the optimal algebra associated to the symmetry group of (2), that shows a systematic way to classify the invariant solutions.

To obtain the optimal algebra, we should first calculate the corresponding commutator table, which can be obtained from the operator

$$[\Pi_\alpha, \Pi_\beta] = \Pi_\alpha \Pi_\beta - \Pi_\beta \Pi_\alpha = \sum_{i=1}^{n} \left(\Pi_\alpha(\xi_\beta^i) - \Pi_\beta(\xi_\alpha^i)\right)\frac{\partial}{\partial x^i}, \quad (9)$$

where $i = 1,2$, with $\alpha, \beta = 1, \cdots, 5$ and $\xi_\alpha^i, \xi_\beta^i$ are the corresponding coefficients of the infinitesimal operators $\Pi_\alpha, \Pi_\beta$. After applying the operator (9) to the symmetry group of (2), we obtain the operators that are shown in the table I.

1.

TABLE I

CONMMUTATORS TABLE ASSOCIATED TO THE SYMMETRY GROUP OF (2).

| [ ; ] | $\boldsymbol{\Pi_1}$ | $\boldsymbol{\Pi_2}$ | $\boldsymbol{\Pi_3}$ | $\boldsymbol{\Pi_4}$ | $\boldsymbol{\Pi_5}$ |
|---|---|---|---|---|---|
| $\boldsymbol{\Pi_1}$ | 0 | 0 | $2\Pi_1$ | 0 | $\Pi_1$ |
| $\boldsymbol{\Pi_2}$ | 0 | 0 | $2\Pi_2$ | $-\Pi_1$ | $\Pi_2$ |
| $\boldsymbol{\Pi_3}$ | $-2\Pi_1$ | $-2\Pi_2$ | 0 | $-2\Pi_2$ | 0 |
| $\boldsymbol{\Pi_4}$ | 0 | $\Pi_1$ | $2\Pi_2$ | 0 | $\Pi_4$ |
| $\boldsymbol{\Pi_5}$ | $-\Pi_1$ | $-\Pi_2$ | 0 | $-\Pi_4$ | 0 |

To proceed, our next task is to compute the adjoint action representation of the symmetries for (2). In order to do so, we will refer to table I, and make use of the relevant operator:

$$Ad\big(exp(\lambda\Pi)\big)G = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\big(ad(\Pi)\big)^n G \text{ for the symmetries } \Pi \text{ and } G. \quad (9.1)$$

Making use of the operator (9.1), we can construct the table II, which shows the adjoint representation for each $\Pi_i$.

TABLE II

ADJOINT REPRESENTATION OF THE SYMMETRY GROUP OF (2).

| Adj[ , ] | $\boldsymbol{\Pi_1}$ | $\boldsymbol{\Pi_2}$ | $\boldsymbol{\Pi_3}$ | $\boldsymbol{\Pi_4}$ | $\boldsymbol{\Pi_5}$ |
|---|---|---|---|---|---|
| $\boldsymbol{\Pi_1}$ | $\Pi_1$ | $\Pi_2$ | $\Pi_3 - 2\lambda\Pi_1$ | $\Pi_4$ | $\Pi_5 - \lambda\Pi_1$ |
| $\boldsymbol{\Pi_2}$ | $\Pi_1$ | $\Pi_2$ | $\Pi_3 - 2\lambda\Pi_2$ | $\Pi_4 + \lambda\Pi_1$ | $\Pi_5 - \lambda\Pi_2$ |
| $\boldsymbol{\Pi_3}$ | $e^{2\lambda}\Pi_1$ | $e^{2\lambda}\Pi_2$ | $\Pi_3$ | $\Pi_4 + 2e^{2\lambda}\Pi_2$ | $\Pi_5$ |
| $\boldsymbol{\Pi_4}$ | $\Pi_1$ | $\Pi_2 - \lambda\Pi_1$ | $\Pi_3 - 2\lambda\Pi_2 + \lambda^2\Pi_1$ | $\Pi_4$ | $\Pi_5 - \lambda\Pi_4$ |
| $\boldsymbol{\Pi_5}$ | $e^{\lambda}\Pi_1$ | $e^{\lambda}\Pi_2$ | $\Pi_3$ | $e^{\lambda}\Pi_4$ | $\Pi_5$ |

Proposition 2. The optimal algebra associated to (2) is given by the vector fields

$$\Pi_1,\ \Pi_2,\ \Pi_4,\ a_2\Pi_2 - \frac{a_1}{a_2}\Pi_4 + \Pi_5,\ \pm\sqrt{-2a_2}\,\Pi_2 + a_3\Pi_3 + \Pi_4,\ -b_{12}\Pi_1 + a_2\Pi_2 + \Pi_4,\ b_{13}^2\Pi_1 - 2b_{13}\Pi_2 + \Pi_3,\ a_1\Pi_1 - b_{11}\Pi_4 + \Pi_5,\ b_4\Pi_1 + b_5\Pi_2 + a_3\Pi_3 + \Pi_5,\ b_9\Pi_1 + a_3\Pi_3 - b_{10}\Pi_4 + \Pi_5. \quad (9a)$$

*Proof*. To calculate the optimal algebra system, we start with the generators of symmetries (4) and a generic nonzero vector. Let

$$G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + a_5\Pi_5. \quad (10)$$

The objective is to simplify as many coefficients $a_i$ as possible, through maps adjoint to $G$, using table II.

Assuming $a_5 = 1$ in (10) we have that $G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Applying the adjoint operator to $(\Pi_5, G)$, we don't have any reduction, on the other hand applying the adjoint operator to $(\Pi_1, G)$ we get

$$G_1 = Ad\big(\exp(\lambda_1\Pi_1)\big)G = \big(a_1 - \lambda_1(1+2a_3)\big)\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5. \quad (11)$$

1.1) Case $1 + 2a_3 \neq 0$. Using $\lambda_1 = \frac{a_1}{1+2a_3}$ with $1 + 2a_3 \neq 0$, in (11), $\Pi_1$ is eliminated, therefore $G_1 = a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Now, applying the adjoint operator to $(\Pi_2, G_1)$, we get $G_2 = Ad\big(\exp(\lambda_2\Pi_2)\big)G_1 = a_4\lambda_2\Pi_1 + (a_2 - \lambda_2(1+2a_3))\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. As $1 + 2a_3 \neq 0$, we can use $\lambda_2 = \frac{a_2}{1+2a_3}$, then is eliminated $\Pi_2$, thus $G_2 = b_1\Pi_1 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$, with $b_1 = \frac{a_4 a_2}{1+2a_3}$. Now, applying the adjoint operator to $(\Pi_3, G_2)$, we get

$$G_3 = Ad\big(\exp(\lambda_3\Pi_3)\big)G_2 = b_1 e^{2\lambda_3}\Pi_1 + 2a_4 e^{2\lambda_3}\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5. \quad (12)$$

It is clear that in (12), we don't have any reduction, then we get $G_3 = b_2\Pi_1 + b_3\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$, with $b_2 = b_1 e^{2\lambda_3}$ and $b_3 = 2a_4 e^{2\lambda_3}$. Now applying the adjoint operator to $(\Pi_4, G_3)$, we get $G_4 = Ad\big(\exp(\lambda_4\Pi_4)\big)G_3 = (b_2 - b_3\lambda_4 + a_3\lambda_4^2)\Pi_1 + (b_3 - 2a_3\lambda_4)\Pi_2 + a_3\Pi_3 + (a_4 - \lambda_4)\Pi_4 + \Pi_5$.

Using $\lambda_4 = a_4$, $\Pi_4$ is eliminated, therefore $G_4 = b_4\Pi_1 + b_5\Pi_2 + a_3\Pi_3 + \Pi_5$, with $b_4 = b_2 - b_3\lambda_4 + a_3\lambda_4^2$ and $b_5 = b_3 - 2a_3\lambda_4$. Then, we have the first element of the optimal algebra

$$G_4 = b_4\Pi_1 + b_5\Pi_2 + a_3\Pi_3 + \Pi_5, \text{ with } a_3 \neq 0. \quad (13)$$

This is how the first reduction of the generic element (10) ends.

1.2) Case $1 + 2a_3 = 0$. We get, $G_1 = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Now, applying the adjoint operator to $(\Pi_2, G_1)$, we have $G_5 = Ad\big(\exp(\lambda_5\Pi_2)\big)G_1 = (a_1 + a_4\lambda_5)\Pi_1 + \big(a_2 - \lambda(1 + 2a_3)\big)\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5 = (a_1 + a_4\lambda_5)\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$.

1.2.$A$) Case $a_4 \neq 0$. Using $\lambda_5 = \frac{-a_1}{a_4}$, then $\Pi_1$ is eliminated, 2. then we get $G_5 = a_2\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. Now applying the adjoint operator to $(\Pi_3, G_5)$, we have $G_6 = Ad\big(\exp(\lambda_6\Pi_3)\big)G_5 = e^{2\lambda_6}(a_2 + 2a_4)\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$. We don't have any reduction, then we get $G_6 = b_6\Pi_2 + a_3\Pi_3 + a_4\Pi_4 + \Pi_5$, with $b_6 = e^{2\lambda_6}(a_2 + a_4)$. Now applying the adjoint operator to $(\Pi_4, G_6)$, we get $G_7 = Ad\big(\exp(\lambda_7\Pi_4)\big)G_6 = (a_3\lambda_7^2 - b_6\lambda_7)\Pi_1 + (b_6 - 2a_3\lambda_7)\Pi_2 + a_3\Pi_3 + (a_4 - \lambda_7)\Pi_4 + \Pi_5$.

Using $\lambda_7 = a_4$, is elimininated $\Pi_4$, then we get $G_7 = b_7\Pi_1 + b_8\Pi_2 + a_3\Pi_3 + \Pi_5$, with $b_7 = a_3\lambda_7^2 - b_6\lambda_7$ and $b_8 = b_6 - 2a_3\lambda_7$. Thus, we have other element of the optimal algebra

$$G_7 = b_7\Pi_1 + b_8\Pi_2 + a_3\Pi_3 + \Pi_5. \quad (14)$$

This is how other reduction of the generic element (10) ends.

1.2.$B$) Case $a_4 = 0$. We get, $G_5 = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + \Pi_5$. Now, applying the adjoint operator to $(\Pi_3, G_5)$, we don't have any reduction, then applying the adjoint operator to $(\Pi_4, G_5)$ we get $G_8 = Ad\big(\exp(\lambda_8\Pi_4)\big)G_5 = (a_1 - a_2\lambda_8 + a_3\lambda_8^2)\Pi_1 + (a_2 - 2a_3\lambda_8)\Pi_2 + a_3\Pi_3 - \lambda_8\Pi_4 + \Pi_5$.

1.2.$B$.1) Case $a_3 \neq 0$. Using $\lambda_8 = \frac{a_2}{2a_3}$, with $a_3 \neq 0$, is eliminated $\Pi_2$, we get $G_8 = \left(a_1 - \frac{a_2^2}{2a_3} + \frac{a_2^2}{4a_3}\right)\Pi_1 + a_3\Pi_3 - \frac{a_2}{2a_3}\Pi_4 + \Pi_5$. Then using $b_9 = a_1 - \frac{a_2^2}{2a_3} + \frac{a_2^2}{4a_3}$ and $b_{10} = \frac{a_2}{2a_3}$, we have other element of the optimal algebra

$$G_8 = b_9\Pi_1 + a_3\Pi_3 - b_{10}\Pi_4 + \Pi_5. \quad (15)$$

This is how other reduction of the generic element (10) ends.

1.2.$B$.2) Case $a_3 = 0$. We get $G_8 = (a_1 - a_2\lambda_8)\Pi_1 + a_2\Pi_2 - \lambda_8\Pi_4 + \Pi_5$.

1.2.$B$.2.$A_1$) Case $a_2 \neq 0$. Using $\lambda_8 = \frac{a_1}{a_2}$, with $a_2 \neq 0$, is eliminated $\Pi_1$, then we have $G_8 = a_2\Pi_2 - \frac{a_1}{a_2}\Pi_4 + \Pi_5$. After, we have other element of the optimal algebra

$$G_8 = a_2\Pi_2 - \frac{a_1}{a_2}\Pi_4 + \Pi_5. \quad (16)$$

This is how other reduction of the generic element (10) ends.

1.2.$B$.2.$A_2$) Case $a_2 = 0$. We have $G_8 = a_1\Pi_1 - \lambda_8\Pi_4 + \Pi_5$. Then using $\lambda_8 = b_{11}$, we have other element of the optimal algebra

$$G_8 = a_1\Pi_1 - b_{11}\Pi_4 + \Pi_5. \quad (17)$$

This is how other reduction of the generic element (10) ends.

Assuming $a_5 = 0$ and $a_4 = 1$ in (10), we have that $G = a_1\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + \Pi_4$. Applying the adjoint operator to $(\Pi_5, G)$ and $(\Pi_3, G)$ we don't have any reduction, on the other hand applying the adjoint operator to $(\Pi_1, G)$ we get

$$G_9 = Ad\big(\exp(\lambda_9\Pi_1)\big)G = (a_1 - 2a_3\lambda_9)\Pi_1 + a_2\Pi_2 + a_3\Pi_3 + \Pi_4. \quad (18)$$

2.1) Case $a_3 \neq 0$, in (18). Using $\lambda_9 = \frac{a_1}{2a_3}$ with $a_3 \neq 0$, in (18), $\Pi_1$ is eliminated, therefore $G_9 = a_2\Pi_2 + a_3\Pi_3 + \Pi_4$. Now, applying the adjoint operator to $(\Pi_2, G_9)$, we get $G_{10} = Ad\big(\exp(\lambda_{10}\Pi_2)\big)G_9 = \lambda_{10}\Pi_1 + (a_2 - 2a_3\lambda_{10})\Pi_2 + a_3\Pi_3 + \Pi_4$. As $a_3 \neq 0$, we can use $\lambda_{10} = \frac{a_2}{2a_3}$, then is eliminated $\Pi_2$, thus we have $G_{10} = \frac{a_2}{2a_3}\Pi_1 + a_3\Pi_3 + \Pi_4$. Now, applying the adjoint operator to $(\Pi_4, G_{10})$, we get $G_{11} = Ad\big(\exp(\lambda_{11}\Pi_4)\big)G_{10} = \left(\frac{a_2}{2a_3} + a_3\lambda_{11}^2\right)\Pi_1 - 2a_3\lambda_{11}\Pi_2 + a_3\Pi_3 + \Pi_4$. Using $\lambda_{11} = \pm\frac{\sqrt{-a_2}}{\sqrt{2}a_3}$, with $a_2 < 0$, is eliminated $\Pi_1$, then we get $G_{11} = \sqrt{-2a_2}\Pi_2 + a_3\Pi_3 + \Pi_4$, with $a_2 > 0$. After, we have other element of the optimal algebra

$$G_{11} = \pm\sqrt{-2a_2}\Pi_2 + a_3\Pi_3 + \Pi_4. \quad (19)$$

with $a_2 < 0$. This is how other reduction of the generic element (10) ends.

2.2) Case $a_3 = 0$, in (18). We get $G_9 = a_1\Pi_1 + a_2\Pi_2 + \Pi_4$. Now, applying the adjoint operator to $(\Pi_2, G_9)$, we have $G_{12} = Ad\big(\exp(\lambda_{12}\Pi_2)\big)G_9 = (a_1 + \lambda_{12})\Pi_1 + a_2\Pi_2 + \Pi_4$, using $\lambda_{12} = -a_1$, is eliminated $\Pi_1$, thus we have $G_{12} = a_2\Pi_2 + \Pi_4$. Now applying the adjoint operator to $(\Pi_4, G_{12})$, we have $G_{13} = Ad\big(\exp(\lambda_{13}\Pi_4)\big)G_{12} = -a_2\lambda_{13}\Pi_1 + a_2\Pi_2 + \Pi_4$.

2.2.$A$) Case $a_2 \neq 0$. Using $\lambda_{13} = \frac{b_{12}}{a_2}$, with $a_2 \neq 0$, we have other element of the optimal algebra

$$G_{13} = -b_{12}\Pi_1 + a_2\Pi_2 + \Pi_4. \quad (20)$$

This is how other reduction of the generic element (10) ends.

$2.2.B)$ Case $a_2 = 0$. We get $G_{13} = \Pi_4$. We have other element of the optimal algebra

$$G_{13} = \Pi_4. \tag{21}$$

This is how other reduction of the generic element (10) ends.

Assuming $a_4 = a_5 = 0$ and $a_3 = 1$ in (10), we have that $G = a_1\Pi_1 + a_2\Pi_2 + \Pi_3$. Applying the adjoint operator to $(\Pi_3, G)$ and $(\Pi_5, G)$ we don't have any reduction, on the other hand applying the adjoint operator to $(\Pi_1, G)$ we get

$$\begin{aligned} G_{14} &= Ad\big(\exp(\lambda_{14}\Pi_1)\big)G \\ &= (a_1 - 2\lambda_{14})\Pi_1 + a_2\Pi_2 + \Pi_3. \end{aligned} \tag{22}$$

Using $\lambda_{14} = \frac{a_1}{2}$, in (22), $\Pi_1$ is eliminated, therefore $G_{14} = a_2\Pi_2 + \Pi_3$. Now, applying the adjoint operator to $(\Pi_2, G_{14})$, we get $G_{15} = Ad\big(\exp(\lambda_{15}\Pi_2)\big)G_{14} = (a_2 - 2\lambda_{15})\Pi_2 + \Pi_3$, using $\lambda_{15} = \frac{a_2}{2}$, is eliminated $\Pi_2$, therefore $G_{15} = \Pi_3$. Now, applying the adjoint operator to $(\Pi_4, G_{15})$, we get $G_{16} = Ad\big(\exp(\lambda_{16}\Pi_4)\big)G_{15} = \lambda_{16}^2\Pi_1 - 2\lambda_{16}\Pi_2 + \Pi_3$. It is clear that we don't have any reduction, then using $\lambda_{16} = b_{13}$, we have other element of the optimal algebra

$$G_{16} = b_{13}^2\Pi_1 - 2b_{13}\Pi_2 + \Pi_3. \tag{23}$$

This is how other reduction of the generic element (10) ends.

Assuming $a_3 = a_4 = a_5 = 0$ and $a_2 = 1$ in (10), we have that $G = a_1\Pi_1 + \Pi_2$. Applying the adjoint operator to $(\Pi_1, G)$, $(\Pi_2, G)$, $(\Pi_3, G)$ and $(\Pi_5, G)$ we don't have any reduction, on the other hand, applying the adjoint operator to $(\Pi_4, G)$ we get

$$G_{17} = Ad\big(\exp(\lambda_{17}\Pi_4)\big)G = (a_1 - \lambda_{17})\Pi_1 + \Pi_2. \tag{24}$$

Using $\lambda_{17} = a_1$, in (24), is eliminated $\Pi_1$, therefore $G_{17} = \Pi_2$. Then, we have other element of the optimal algebra

$$G_{17} = \Pi_2. \tag{25}$$

This is how other reduction of the generic element (10) ends.

Assuming $a_2 = a_3 = a_4 = a_5 = 0$ and $a_1 = 1$ in (10), we have that $G = \Pi_1$. Applying the adjoint operator to $(\Pi_1, G)$, $(\Pi_2, G)$, $(\Pi_3, G)$, $(\Pi_4, G)$ and $(\Pi_5, G)$ we don't have any reduction, then we have other element of the optimal algebra

$$G = \Pi_1. \tag{26}$$

This is how other reduction of the generic element (10) ends.

Taking into account the reductions presented in (13), (14), (15), (16), (17), (19), (20), (21), (23), (25), and (26), we have all the elements of the optimal system presented in Proposition 2, (9a). (Note that reductions (13) and (14) are essentially the same). □

## IV. Invariant Solutions through Optimal Algebra Generators

In this section, we characterize all invariant solutions taking into account some operators that generate the optimal algebra presented in Proposition 2. For this purpose, we use the method of invariant curve condition [24] (presented in section 4.3), which is given by the following equation

$$Q(x, y, y_x) = \eta - y_x\xi = 0. \tag{27a}$$

Using the element $\Pi_1$ from Proposition 2, under the condition (27a), we obtain that $Q = \eta_1 - y_x\xi_1 = 0$, which implies $(0) - y_x(-(x+y)^{-1}) = 0$, then solving this ODE we have $y(x) = c$, which is the trivial solution for (2), using an analogous procedure with all of the elements of the optimal algebra (Proposition 2), we obtain both implicit and explicit invariant solutions that are shown in the table III, with $c$ being a constant.

TABLE III

SOLUTIONS FOR (2) USING INVARIANT CURVE CONDITION.

| № | Elements | $Q(x,y,y_x) = 0$ | Solutions | Type Solution |
|---|---|---|---|---|
| 1 | $\Pi_1$ | $(0) - y_x(-(x+y)^{-1}) = 0$ | $y(x) = c$ | Trivial |
| 2 | $\Pi_2$ | $(0) - y_x(-y(x+y)^{-1}) = 0$ | $y(x) = 0,\ y(x) = c$ | Trivial |
| 3 | $\Pi_4$ | $(1) - y_x(-1) = 0$ | $y(x) = c - x$ | Explicit |
| 4 | $\Pi_2 - \Pi_4 + \Pi_5$ | $(y-1) - y_x(x + 1 - y(x+y)^{-1}) = 0$ | $y(x) = \frac{1}{x\left(\pm\frac{1}{\sqrt{x}(x+2)^{3/2}\sqrt{c - \frac{2}{x+2}}} - \frac{1}{-x^2-2x}\right) - 1} - x$ | Explicit |
| 5 | $-\sqrt{2}\Pi_2 + \Pi_3 + \Pi_4$ | $(1) - y_x\left(-1 + (x+y)^{-1}\left(\sqrt{2}y + x^2 + 2xy\right)\right) = 0$ | $---$ | $-$ |
| 6 | $-\Pi_1 + \Pi_2 + \Pi_4$ | $(1) - y_x\left(-1 + (x+y)^{-1}(1-y)\right) = 0$ | $y(x) = \frac{1-x}{2} - \frac{\sqrt{c - x^2 + \frac{1}{2}(x-1)^2}}{\sqrt{2}},\ y(x) = \frac{\sqrt{c - x^2 + \frac{1}{2}(x-1)^2}}{\sqrt{2}} + \frac{1-x}{2}$ | Explicit |
| 7 | $\Pi_1 - 2\Pi_2 + \Pi_3$ | $(0) - y_x\left((x+y)^{-1}(-1 + 2y + x^2 + 2xy)\right) = 0$ | $y(x) = \frac{1-x}{2},\ y(x) = c$ | Explicit |
| 8 | $\Pi_1 - \Pi_4 + \Pi_5$ | $(y-1) - y_x(x + 1 - (x+y)^{-1}) = 0$ | $y(x) = \frac{1}{(x+1)\left(\frac{1}{x^2+2x} \pm \frac{x+1}{(1-(x+1)^2)^{3/2}\sqrt{c - \frac{2}{2x^2+4x}}}\right)} - \frac{x^2+x-1}{x+1}$ | Explicit |
| 9 | $\Pi_1 + \Pi_2 + \Pi_3 + \Pi_5$ | $(y) - y_x\left(x + (x+y)^{-1}(-1 - y + x^2 + 2xy)\right) = 0$ | $---$ | $-$ |
| 10 | $\Pi_1 + \Pi_3 - \Pi_4 + \Pi_5$ | $(y-1) - y_x\left(x + 1 + (x+y)^{-1}(x^2 + 2xy - 1)\right) = 0$ | $---$ | $-$ |

## V. Exploring Conserved Quantities and Variational Symmetries

In this section, we shall introduce the variational symmetries associated with (2) and employ them to define conservation laws based on Noether's theorem [2] (p. 235–267). Our first task is to identify the Lagrangian by applying the Jacobi Last Multiplier technique, which was first introduced by Nucci [27]. Consequently, it becomes necessary to evaluate the inverse of the determinant $\Delta$:

$$\Delta = \begin{vmatrix} x & y_x & y_{xx} \\ \Pi_{4,x} & \Pi_{4,y} & \Pi_4^{(1)} \\ \Pi_{5,x} & \Pi_{5,y} & \Pi_5^{(1)} \end{vmatrix} = \begin{vmatrix} x & y_x & y_{xx} \\ -1 & 1 & 0 \\ x & y & 0 \end{vmatrix}, \tag{27b}$$

where $\Pi_{4,x}, \Pi_{4,y}, \Pi_{5,x}$, and $\Pi_{5,y}$ are the components of the symmetries $\Pi_4, \Pi_5$ shown in the Proposition 1 and $\Pi_4^{(1)}, \Pi_5^{(1)}$ as its first prolongations. Then we get in (27b), $\varDelta = -(x+y)$ which implies that $M = \frac{1}{\varDelta} = -(x+y)^{-1}$. Now, from [27], we know that $M$ can also be written as $M = L_{y_x y_x}$ which means that $L_{y_x y_x} = -(x+y)^{-1}$, then integrating twice with respect to $y_x$ we obtain the Lagrangian

$$L(x,y,y_x) = -\frac{y_x^2(x+y)^{-1}}{2} + y_x f_1(x,y) + f_2(x,y), \quad (28)$$

where $f_1, f_2$ are arbitrary functions. From the preceding expression we can consider $f_1 = f_2 = 0$. It's possible to find more Lagrangians for (2) by considering other vector fields given in the Proposition 1. Then, we calculate

$$\xi(x,y)L_x + \xi_x(x,y)L + \eta(x,y)L_y + \eta_{[x]}(x,y)L_{y_x} = D_x[f(x,y)], \quad (28a)$$

To calculate (28a), we use (28) and (7a). Thus we get

$$\xi\left(\frac{y_x^2(x+y)^{-2}}{2}\right) + \xi_x\left(-\frac{y_x^2(x+y)^{-1}}{2}\right) + \eta\left(\frac{y_x^2(x+y)^{-2}}{2}\right) + \left(\eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2\right)(-y_x(x+y)^{-1}) - f_x - y_x f_y = 0. \quad (28b)$$

In (28b), rearranging and associating terms with respect to $1, y_x, y_x^2$ and $y_x^3$, we obtain the following determinant equations

$$\begin{aligned} \xi_y = f_x &= 0, && (29a)\\ (x+y)^{-1}\eta_x + f_y &= 0, && (29b)\\ \xi - \xi_x(x+y) + \eta - 2(x+y)(\eta_y - \xi_x) &= 0. && (29c) \end{aligned}$$

Solving the preceding system (29a, 29b, 29c) for $\xi, \eta$, and $f$, we obtain the infinitesimal generators of Noether's symmetries

$$\eta = -\frac{x^2}{2}a_1 - xya_1 + ya_2 - a_3, \qquad \xi = -\frac{x^2}{2}a_1 + a_2 x + a_3,$$
$$\text{and } f(y) = ya_1 + a_4. \quad (30)$$

with $a_1, a_2, a_3$ and $a_4$ arbitrary constants. Then, the Noether symmetry group or variational symmetries are

$$V_1 = -\frac{x^2}{2}\frac{\partial}{\partial x} - \left(\frac{x^2}{2} + xy\right)\frac{\partial}{\partial y}, \qquad V_2 = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial y}, \qquad V_3 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}. \quad (31)$$

Remarks: Note that in (31), $V_2 = \varPi_5$ and $V_3 = -\varPi_4$ thus, the symmetries in (2) have two variational symmetry. According to [28], in order to obtain the conserved quantities or conservation laws, we should solve

$$I = (Xy_x - Y)L_{y_x} - XL + f, \quad (32)$$

so, using (28), (30) and (31). Therefore, the conserved quantities are given by

$$\begin{aligned} I_1 &= \frac{y_x^2 x^2(x+y)^{-1}}{4} - y_x(x+y)^{-1}\left(\frac{x^2}{2} + xy\right) + ya_1 + a_4,\\ I_2 &= -\frac{y_x^2 x(x+y)^{-1}}{2} + yy_x(x+y)^{-1} + ya_1 + a_4,\\ I_3 &= -\frac{y_x^2(x+y)^{-1}}{2} - y_x(x+y)^{-1} + ya_1 + a_4. \quad (33) \end{aligned}$$

## VI. Categorization of Lie algebra

In the realm of finite dimensional Lie algebras over a field of characteristic 0, Levi's theorem provides a generic classification scheme. Specifically, the theorem asserts that every finite dimensional Lie algebra can be expressed as a semidirect product of a solvable Lie algebra and a semisimple Lie algebra, with the solvable Lie algebra serving as the radical of the overall algebra. This implies the existence of two major classes of Lie algebras, namely the solvable and the semisimple. Notably, each of these classes features subclasses that possess distinct classifications. For instance, within the solvable class, we encounter the nilpotent Lie algebra.

Based on the Lie group symmetries generated as outlined in table I, a Lie algebra with five dimensions can be derived. In order to classify this Lie algebra, we must first recall certain fundamental criteria. Specifically, in the case of solvable and semisimple Lie algebras, the Cartan-Killing form K(.,.) serves as an important tool for classification. Further details and propositions relating to these criteria can be found in [29].

**P**roposition 3. (Cartan's theorem) If and only if the Killing form of a Lie algebra is nondegenerate, it is classified as semisimple.

Proposition 4. The solvability of a Lie subalgebra $\mathfrak{g}$ can be determined by evaluating $K(X,Y)$ for all $X \in [\mathfrak{g},\mathfrak{g}]$ and $Y \in \mathfrak{g}$. Specifically, if $K(X,Y) = 0$ for all such $X$ and $Y$, then $\mathfrak{g}$ is classified as solvable. An alternative representation of this statement is $K(\mathfrak{g},[\mathfrak{g},\mathfrak{g}]) = 0$.

The next statements are imperative for the successful completion of the classification.

Definition 1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field $k$. Choose a basis $e_j$, $1 \le i \le n$, in $\mathfrak{g}$ where $n = \dim \mathfrak{g}$ and set $[e_i, e_j] = C_{ij}^k e_k$. The coefficients $C_{ij}^k$ are denoted as the structure constants.

Proposition 5. Consider two Lie algebras, $\mathfrak{g}_1$ and $\mathfrak{g}_2$, each with a dimension of $n$. Assuming that each algebra has a basis such that the structure constants are identical, it follows that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.

Consider the Lie algebra $\mathfrak{g}$, which corresponds to the symmetry group of infinitesimal generators of (1) as specified in the commutator table. In order to proceed, it suffices to examine the following relations:

$$[\varPi_1,\varPi_3] = 2\varPi_1,\ [\varPi_1,\varPi_5] = \varPi_1,\ [\varPi_2,\varPi_3] = 2\varPi_2,\ [\varPi_2,\varPi_4] = -\varPi_1,\ [\varPi_2,\varPi_5] = \varPi_2,\ [\varPi_3,\varPi_4] = -2\varPi_2,\ [\varPi_4,\varPi_5] = \varPi_4. \quad (34)$$

Using (34), we calculate Cartan-Killing form $K$ as follows.

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 8 & 0 & 4\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 4 & 0 & 3 \end{bmatrix}, \quad (35)$$

In (35) the determinant vanishes, and thus by Proposition 3 (Cartan criterion) it is not semisimple. Given that the Cartan-Killing form of a nilpotent Lie algebra is uniformly zero, it follows that the Lie algebra $\mathfrak{g}$ is not nilpotent, by the converse of the preceding assertion. Next, we confirm the solvability of the Lie algebra by applying the Cartan criteria for solvability, as established in Proposition 4. Consequently, we are dealing with a non-nilpotent, but solvable Lie algebra. The Nilradical of $\mathfrak{g}$ is generated by $\Pi_1, \Pi_2$, and $\Pi_4$, that is, we have a Solvable Lie algebra with three dimensional Nilradical.

Let $m$ the dimension of the Nilradical $M$ of a Solvable Lie algebra, In this case, in fifth dimensional Lie algebra we have that $3 \leq m \leq 5$. Mubarakzyanov in [30] classified the 5-dimensional solvable nonlilpotent Lie algebras, in particular the solvable nonnilpotent Lie algebra with three dimensional Nilradical, this Nilradical is isomorphic to $\mathfrak{h}_3$ the Heisenberg Lie algebra. Then, by the Proposition 5, and consequently we establish a isomorphism of Lie algebras with $\mathfrak{g}$ and the Lie algebra $\mathfrak{g}_{5,40}$. In summery we have the next proposition.

Let us consider the dimension of the Nilradical, denoted as $M$, of a Solvable Lie algebra, which we denote as $m$. In the case of a fifth-dimensional Lie algebra, we can establish that $3 \leq m \leq 5$. In [30], Mubarakzyanov classified the 5-dimensional solvable non-nilpotent Lie algebras, including the solvable non-nilpotent Lie algebra with a three-dimensional Nilradical. Notably, this Nilradical is isomorphic to $\mathfrak{h}_3$, the Heisenberg Lie algebra. Based on Proposition 5, we can establish an isomorphism of Lie algebras between $\mathfrak{g}$ and the Lie algebra $\mathfrak{g}_{5,40}$. In summary, we have the following proposition.

Proposition 6. The 5-dimensional Lie algebra $\mathfrak{g}$, which is associated with the symmetry group of (1), is a solvable non-nilpotent Lie algebra possessing a three-dimensional Nilradical. In the classification presented by Mubarakzyanov, it is shown that the Lie algebra discussed above is isomorphic to $\mathfrak{g}_{5,40}$.

## VII. Conclusion

Using the Lie symmetry group (see Proposition 1), we calculated the optimal algebra, as it was presented in Proposition 2. Using these operators it was possible to characterize all the invariant solutions (see table III), these solutions are different from (3) and these solutions do not appear in the literature known until today. It has been shown the variational symmetries for (2) in (31) with its corresponding conservation laws (33) which fulfills condition (32). The Lie algebra associated to (2) is a solvable nonnilpotent Lie algebra with three dimensional Nilridical. Besides that Lie algebra is isomorphich with $\mathfrak{g}_{5,34}$ in the Mubarakzyanov's classification.

The findings of this study are novel and consistent with the underlying principles governing these equations, which have extensive applications in various scientific fields. As such, they have the potential to be of significant importance to many researchers. The ultimate goal of this investigation was successfully achieved by demonstrating the relevance of these results to the broader scientific community, and by providing insights into the fundamental mechanisms underlying these phenomena.

For future works, equivalence group theory could be also considered to obtain preliminary classifications associated to a complete classification of (2).

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## Declaration interests

The authors declare that they have no conflict of interest.

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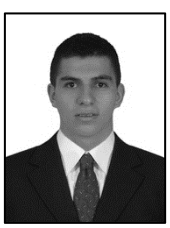
**Gabriel Ignacio Loaiza Ossa**. PhD D. in Mathematical Sciences, Universidad Politécnica de Valencia, Spain. Full-time professor at Universidad EAFIT, Colombia.
ORCID: https://orcid.org/0000-0003-2413-1139



**Yeisson Alexis Acevedo Agudelo**. Master's Degree in Applied Mathematics, Universidad EAFIT, Colombia.
ORCID: https://orcid.org/0000-0002-1640-9084

**Oscar Mario Londoño Duque**. PhD of applied mathematics, University of Campinas, Brazil.
ORCID: https://orcid.org/0000-0002-5666-8224

**Danilo A. García Hernández**. PhD in Mathematics, University of Campinas, Brazil.
ORCID: https://orcid.org/0000-0002-0807-2602